ON A CONSTRUCTIVE DEFINITION OF THE RESTRICTED DENJOY INTEGRAL

BY

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1. **Introduction.** Let $I=[a, b]$ and F be a function of the closed subintervals of I. One says that F has Burkill integral $\int_J F$ over the closed interval $J \subseteq I$ if

$$\int_J F = \lim_{P \in \mathscr{P}: |P| \to 0} \sum_{I' \in P} F(I'),$$

where $\mathscr{P}$ is the set of partitions of J and $|P|$ is the norm of P. The Burkill integral has been employed in a more general setting [1] to give a descriptive definition of the restricted Denjoy integral of point functions f. In this paper we show how this integral can be used to give a constructive definition of the restricted Denjoy integral and compare the classical construction with ours. We adopt the convention that I and J, with or without subscripts or superscripts, always denote a closed interval.

Before we begin our discussion, let us recall the classical constructive definition ([2], 255–259) of the restricted Denjoy integral.

Let T be a real-valued function whose domain, dom T, is a set of ordered pairs $\{(f, J)\}$, where f is a real-valued point function defined on J. The set

$$\{f: (f, J) \in \text{dom } T\}$$

will be denoted by $\text{dom}_J\, T$.

T is called an *integral* if

(i) $f \in \text{dom}_J\, T$ implies $f \in \text{dom}_{J'}\, T$ for all closed subintervals $J' \subseteq J$, and $T(f, J')$ is an additive, continuous function of J';

(ii) if $f \in \text{dom}_{I_i}\, T$, $i=1, 2$, where I_1 and I_2 are abutting, then $f \in \text{dom}_{I_1 \cup I_2}\, T$;

(iii) if $f \equiv 0$ on I, then $f \in \text{dom}_I\, T$ and $T(f, I)=0$.

One says that f is T-integrable on I if $(f, I) \in \text{dom } T$.

Two integrals T_1 and T_2 are *compatible* if $T_1(f, I')=T_2(f, I')$ whenever they both exist. We say $T_1 \subseteq T_2$ if T_1 and T_2 are compatible and dom $T_1 \subseteq$ dom T_2.

Given a function f defined on $I' \subseteq I$ and an integral T, one says that a point $x \in I'$ is a *T-singular point* of f in I' if there exists $\{I_n\}$, with $I_n \subseteq I'$, $|I_n| \to 0$, $x \in I_n$, such that $(f, I_n) \notin \text{dom } T$.

If $\mathscr{S}$ is the set of T-singular points in I, clearly $\mathscr{S}$ is closed and $(f, I') \in \text{dom } T$ for all $I' \subseteq I$ such that $I' \cap \mathscr{S} = \varnothing$.

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Let T be an integral. One defines $\mathrm{dom}_I\, T^C$ by the following conditions:

(c_1) $\mathscr{S} \cap I$ is finite or void;

(c_2) there exists a continuous, additive F such that $F(I')=T(f, I')$ whenever $(f, I') \in \mathrm{dom}\, T$, $I' \subseteq I$.

Define $T^C(f, I)=F(I)$. It is clear that T^C is an integral.

Let $E \subseteq I$. Let $f_E = f\chi_E$, where χ_E is the characteristic function of E. One says f is T-integrable on E if f_E is T-integrable on I.

One defines $\mathrm{dom}_I\, T^H$ by the following conditions:

(h_1) f is T-integrable on $\mathscr{S}$ and on each of the intervals I_k contiguous to $\mathscr{S} \cup \{a, b\}$.

(h_2) $\sum O(T; f; I_k) < \infty$, where $O(T; f; J) = \sup_{J' \subseteq J} |T(f, J')|$.

Define $T^H(f, I) = T(f, \mathscr{S}) + \sum T(f, I_k)$. Clearly T^H is also an integral.

Let $\{T^\alpha\}$ be a sequence of integrals, in general transfinite, such that $T^\alpha \subseteq T^\beta$ whenever $\alpha < \beta$. Define $\mathrm{dom} \sum_{\beta<\alpha} T^\beta = \bigcup_{\beta<\alpha} \mathrm{dom}\, T^\beta$ and if $(f, I') \in \mathrm{dom} \sum_{\beta<\alpha} T^\beta$, define $(\sum_{\beta<\alpha} T^\beta)(f, I') = T^{\beta_0}(f, I')$, where β_0 is the least ordinal such that $(f, I') \in \mathrm{dom}\, T^{\beta_0}$.

Write $T^{CH} = (T^C)^H$. We define a transfinite sequence $\{D^\alpha\}$ of integrals as follows: let $\mathscr{L}$ be the Lebesgue integral,

$$D^0 = \mathscr{L},$$

$$D^\alpha = \Big(\sum_{\beta<\alpha} D^\beta\Big)^{CH}$$

if $\alpha > 0$. Let Ω be the first uncountable ordinal. Then it is well known ([2], 258) that if D_* is the restricted Denjoy integral,

$$D_* = \sum_{\alpha<\Omega} D^\alpha.$$

2. **A constructive definition using the Burkill integral.** We define $\mathrm{dom}_I\, T^{H^*}$ by the following conditions:

(h_1^*) there exists a closed set $W \subseteq I$ such that f is T-integrable on W and on each $I' \subseteq I$ with $I' \cap W = \varnothing$;

(h_2^*) if we define

$$\psi(I') = T(f, I') \quad \text{if } I' \cap W = \varnothing,$$
$$= 0 \quad\quad\quad \text{if } I' \cap W \neq \varnothing,$$

then $\int_I \psi$ exists and $\int \psi$ is continuous (note that if $\int_I F$ exists, then $\int_{I'} F$ exists for all $I' \subseteq I$ and $\int_{I'} F$ is additive).

Define $T^{H^*}(f, I) = T(f_W, I) + \int_I \psi$. We note that T^{H^*} is an integral. For suppose that $f \in \mathrm{dom}_J\, T^{H^*}$. Let $J' \subseteq J$. If $J' \cap W = \varnothing$, then $f \in \mathrm{dom}_{J'}\, T$ so that, taking $W' = J'$ in the definition of $\mathrm{dom}_{J'}\, T^{H^*}$, we see that $f \in \mathrm{dom}_{J'}\, T^{H^*}$. If $J' \cap W \neq \varnothing$, then since $f_W \in \mathrm{dom}_J\, T$ and T is an integral, $f_W \in \mathrm{dom}_{J'}\, T$. Now, since $\int_J \psi$ exists, so does $\int_{J'} \psi$ ([1], p. 70). Recall that continuity of $\int \psi$ is assumed in the definition of $\text{dom}_J T^{H^*}$. Therefore, in either case $f \in \text{dom}_{J'} T^{H^*}$. Since T is an integral, $T(f_W, J')$ is an additive, continuous function of the $J' \subseteq J$; it is known ([1], p. 70) that $\int \psi$ is additive. Therefore $T^{H^*}(f, J')$ is an additive, continuous function of the $J' \subseteq J$, and condition (i) is satisfied. Now suppose $f \in \text{dom}_{I_i} T^{H^*}$, $i=1, 2$, where I_1 and I_2 are abutting. With no loss in generality we may assume I_1 is to the left of I_2. Choose $W_i \subseteq I_i$, $i=1, 2$, closed sets satisfying the requirements of the definition of $f \in \text{dom}_{I_i} T^{H^*}$, and let ψ_i be the functions corresponding to T^{H^*}, f and W_i, $i=1, 2$, in this definition. Set $a_1 = \sup W_1$, $b_1 = \inf W_2$. If $a_1 = b_1$, choose

$$W = W_1 \cup W_2.$$

Then W is closed and, since T is an integral, by condition (ii), $f_W \in \text{dom}_{I_1 \cup I_2} T$. Let $\varepsilon > 0$ and choose $\eta_i = \eta_i(\varepsilon) > 0$, $i=1, 2$, so that if P_i is a partition of I_i, $i=1, 2$, with $|P_i| < \eta_i$, then $|\sum_{I' \in P_i} \psi_i(I') - \int_{I_i} \psi_i| < \varepsilon/8$, and also so that if $\{J_1, \ldots, J_p\}$ is any finite sequence of nonoverlapping subintervals of I_1 with Max $|J_k| < \eta_i$, then $\sum_1^p |\psi_i(J_k) - \int_{J_k} \psi_i| < \varepsilon/8$ (see [1], p. 70). Also, since $\int \psi_i$ is additive and continuous on I_k, $i=1, 2$, we may choose η, $0 < \eta < \min(\eta_1, \eta_2)$, so that $J \subseteq I_i$, $|J| < \eta$ imply that $|\int_J \psi_i| < \varepsilon/8$. Now let P be any partition of $I_1 \cup I_2$ with $|P| < \eta$. Then

$$\left| \sum_{I' \in P} \psi(I') - \left(\int_{I_1} \psi_1 + \int_{I_2} \psi_2 \right) \right| < \varepsilon.$$

It therefore follows that $\int_{I_1 \cup I_2} \psi = \int_{I_1} \psi_1 + \int_{I_2} \psi_2$. It is immediate that $\int \psi$ is continuous on $I_1 \cup I_2$. If $a_1 < b_1$, choose $W = W_1 \cup W_2 \cup [a_1, b_1]$. A similar argument shows that, again, condition (ii) is satisfied. Finally, if $f \equiv 0$ on I, then since T is an integral, $f \in \text{dom}_I T$ and $T(f, I) = 0$. Choosing $W = I$, we see that $f \in \text{dom}_I T^{H^*}$ and $T^{H^*}(f, I) = 0$. Therefore T^{H^*} is an integral. By a known result ([1], 90), if $T \subseteq D_*$, then $T^{H^*} \subseteq D_*$.

Romanovski ([1], 91) has defined the following sequence $\{K_\alpha\}$ of classes of D_*-integrable functions defined on I. Put $f \in K_0$ if f is Lebesgue integrable on I. Let $\alpha > 0$. Then we put $f \in K_\alpha$ if one of the following four conditions is satisfied:

(1) there is a partition P such that f is in a class K_β, with $\beta < \alpha$, on each member of P;

(2) f can be extended to a function which belongs to a class K_β, with $\beta < \alpha$, on some $I' \supseteq I$;

(3) f belongs to a class K_β, with $\beta < \alpha$, on each closed sub-interval of $I^0 = (a, b)$;

(4) there is a closed set W such that f_W is Lebesgue integrable on I, and such that, on each $I' \subseteq I$ with $I' \cap W = \varnothing$, f belongs to a class K_β, with $\beta < \alpha$; in addition, if

$$\psi(I') = (D_*) \int_{I'} f\,dx \quad \text{if } I' \cap W = \varnothing,$$
$$= 0 \qquad\qquad \text{if } I' \cap W \neq \varnothing,$$

then $\int_I \psi$ exists and $\int \psi$ is continuous. It is known ([1], 91) that if f is D_*-integrable, then f belongs to some class K_α.

We define a transfinite sequence $\{D_*^\alpha\}$ of integrals as follows:

$$D_*^0 = \mathscr{L},$$

$$D_*^\alpha = \left(\sum_{\beta<\alpha} D_*^\beta\right)^{H^*}$$

if $\alpha > 0$.

THEOREM 1. $K_\alpha \subseteq \mathrm{dom}_I\, D_*^{\alpha+1}$.

Proof. Clearly $\mathrm{dom}_I\, D_*^0 = K_0$. Suppose the theorem is true for all $\beta<\alpha$, for some $\alpha>0$ and that $f\in K_\alpha$. If condition (1) of the definition of K_α is satisfied, then there is a partition P of I such that $f\in \mathrm{dom}_{I'}\, D_*^{\beta+1}$ for all $I'\in P$, where $\beta<\alpha$. By condition (ii) for an integral, $f\in\mathrm{dom}_I\, D_*^{\beta+1}$. Since $\beta+1<\alpha+1$, $f\in\mathrm{dom}_I\, D_*^{\alpha+1}$. If condition (2) of the definition of K_α is satisfied, then there exists $I'\supseteq I$ such that $f\in K_\beta$ on I', $\beta<\alpha$. But then by the induction hypothesis, $f\in\mathrm{dom}_{I'}\, D_*^{\beta+1}$ and by condition (i) for integrals, $f\in\mathrm{dom}_I\, D_*^{\beta+1}$. Since $\beta+1<\alpha+1$, again $f\in\mathrm{dom}_I\, D_*^{\alpha+1}$. Now suppose condition (4) of the definition of K_α is satisfied. Then there is a closed set $W\subseteq I$ such that $f_W\in\mathrm{dom}_I\,\mathscr{L}$, and such that, on each $I'\subseteq I$ with $I'\cap W=\varnothing$, $f\in\mathrm{dom}_{I'}\, D_*^{\beta+1}$, where $\beta<\alpha$; in addition, $\int_I \psi$ exists and $\int\psi$ is continuous. It follows that $f\in\mathrm{dom}_I\, D_*^{\alpha+1}$.

We finally show that if condition (3) of the definition is satisfied, then f is $D_*^{\alpha+1}$-integrable. Now $f\in\mathrm{dom}_{I'}\, D_*^\beta$, with $\beta<\alpha+1$, for all $I'\subseteq(a,b)$. Let

$$F = (D_*)\int f\,dx.$$

Then F is continuous on I. In (h_1^*) and (h_2^*) choose $T=D_*^\alpha$. Note that $F(I')=T(f,I')$ for all $I'\subseteq I^0$. Let $W=\{a,b\}$ and $\varepsilon>0$. Choose $\delta>0$ such that $I'\subseteq I$, $|I-I'|<\delta$ imply $|F(I)-F(I')|<\varepsilon$. Let P be any partition of I with $|P|<\delta/2$. Then

$$\left|\sum_{I'\in P}\psi(I')-F(I)\right| = \left|\sum_{I'\in P'}F(I')-F(I)\right| < \varepsilon,$$

where P' is the set of those members of P which contain neither a nor b. It follows that $\int_I\psi=F(I)$. Clearly $\int\psi$ is continuous. Therefore

$$K_\alpha \subseteq \mathrm{dom}\, D_*^{\alpha+1}.$$

THEOREM 2. $D_*=\sum_{\alpha<\Omega} D_*^\alpha$.

Proof. Let f be D_*-integrable on I. Define $f=0$ outside of I. Let S^α denote the set of D_*^α-singular points of f in I. Then $\{S^\alpha\}$ is decreasing and, thus, stationary (i.e., there is an $\alpha_0<\Omega$ such that $S^\alpha=S^{\alpha_0}$ for all $\alpha>\alpha_0$; see, e.g., [2], 258). However, $f\in K_\beta$ for some β and, therefore, $f\in\mathrm{dom}_I\, D_*^{\beta+1}$. Thus $S^{\beta+1}=\varnothing$, which implies $S^{\alpha_0}=\varnothing$ and $f\in\mathrm{dom}_I\, D_*^{\alpha_0}$. We have observed above that $T\subseteq D_*$ implies $T^{H^*}\subseteq D_*$. To show that $D_*\supseteq\sum_{\alpha<\Omega} D_*^\beta$ it is easy to see that if $D_*^\beta\subseteq D_*$ for all $\beta<\alpha$, then $\sum_{\beta<\alpha} D_*^\beta\subseteq D_*$, and that therefore $(\sum_{\beta<\alpha} D_*^\beta)^{H^*}\subseteq D_*$. Since $D_*^0=\mathscr{L}\subseteq D_*$, we have the desired containment relation by applying transfinite induction.

3. **Comparison of our construction with the classical construction.** In the following theorem we restrict our attention to those integrals T which have the property that $f \in \text{dom}_I T$ whenever $I \cap \{f \neq 0\}$ is finite.

THEOREM 3. $T^C \subseteq T^{H^*}$.

Proof. Let f be T^C-integrable on I. Let $\mathscr{S}$ be the set of T-singular points in I and $\{I_1, \ldots, I_k\}$ be the set of intervals contiguous to $\mathscr{S} \cup \{a, b\} = E$. Let $F(I') = T^C(f, I')$ for all $I' \subseteq I$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that for each i, $I' \subseteq I_i$ and $|I_i - I'| < \delta$ imply $|F(I_i) - F(I')| < \varepsilon/k$. Let $S_I = \{J_1, \ldots, J_m\}$ be any partition of I with $|S_I| < \delta/2$. Define

$$\begin{aligned} \psi(I') &= F(I') \quad \text{if } I' \cap E = \varnothing, \\ &= 0 \qquad\quad \text{if } I' \cap E \neq \varnothing, \end{aligned}$$

for all $I' \subseteq I$. If $S \subseteq S_I$ is the subset of S_I consisting of those J_i such that $J_i \cap E = \varnothing$, we see that, since F is additive,

$$\left| \sum_{i=1}^{m} \psi(J_i) - F(I) \right| = \left| \sum_{J_i \in S} \psi(J_i) - \sum F(I_j) \right| < \varepsilon.$$

It follows that $\int_I \psi = F(I)$.

We shall show below that the containment guaranteed by Theorem 3 is proper in general, i.e., there are T^{H^*}-integrable functions which are not T^C-integrable.

We note that (h_1^*) and (h_2^*) do not require that f be T-integrable on the intervals I_j contiguous to W. Since $\int_I \psi$ exists, $\int \psi$ is continuous, and $T(f, I')$ is an additive function of the closed subintervals of I_j^0 for all j, it follows that $\int_{I'} \psi = T(f, I')$ for all such I', and therefore f is T^C-integrable on I_j for all j. Moreover,

$$\int_{I_j} \psi = \lim_{|I_j - I'| \to 0} T(f, I').$$

The following theorem will immediately establish the inclusion relationships between T^H, T^{H^*}, and T^{CH}.

THEOREM 4. *Suppose $E \subseteq I$ is closed. Let $\{I_k\}$ be the sequence of intervals contiguous to $E \cup \{a, b\}$. Consider the following two conditions*:

(1) *for every k, $f \in \text{dom}_{I'} T$ for all $I' \subseteq I_k^0$ and $f \in \text{dom}_{I_k} T^C$; moreover, $\sum O(T^C; f; I_k) < \infty$*;

(2) *f is T-integrable on all $I' \subseteq I$ such that $I' \cap E = \varnothing$; moreover, if*

$$\begin{aligned} \psi(I') &= T(f, I') \quad \text{if } I' \cap E = \varnothing, \\ &= 0 \qquad\qquad \text{if } I' \cap E \neq \varnothing, \end{aligned}$$

then $\int_I \psi$ exists and $\int \psi$ is continuous. Then (1) *and* (2) *are equivalent.*

COROLLARY 5. $T^H \subseteq T^{H^*}$.

COROLLARY 6. $T^{H^*} \subseteq T^{CH}$.

These corollaries follow from a comparison of the conditions in the definitions of T^H, T^{H^*}, and T^{CH} with conditions (1) and (2) of Theorem 4.

Proof of Theorem 4. We show first that (1) implies (2). Let $\varepsilon > 0$. Choose $N = N(\varepsilon)$ such that

$$\sum_{N+1}^{\infty} O(T^C; f; I_k) < \varepsilon/3.$$

Let $I_k = [a_k, b_k]$ for all k. Consider I_j with $j \leqq N$. By continuity of $T^C(f, I')$ there is a $\delta_j = \delta_j(\varepsilon) > 0$ such that for $I' \subseteq I_j$, $|T^C(f, I_j) - T^C(f, I')| < \varepsilon/6N$ whenever $|I_j - I'| < \delta_j$. In particular, then, if $J' = [a_j, b']$ and $J'' = [a'', b_j]$ are contained in J_j, and if $b' - a_j < \delta_j$ and $b_j - a'' < \delta_j$, then

$$|T^C(f, J')| + |T^C(f, J'')| < \varepsilon/3N.$$

Note, also, that since $T^C(f, I)$ is additive, if $J' \subseteq I_j$ and $\{J_1', \dots, J_p'\}$ is a partition of J', then $\sum T^C(f, J_i') = T^C(f, J')$.

Now choose $\delta = \min_{j=1,\dots,N} \delta_j$. Suppose $S = \{J_1, \dots, J_p\}$ is a partition of I with $|S| < \delta$. The only members J_i of S for which it is possible that $\psi(J_i) \neq 0$ are those for which $J_i \cap E = \varnothing$. Consider I_k with $k \leqq N$. Let $\{J_1^k, \dots, J_{n(k)}^k\}$ be those J_i which are contained in I_k, such that $J_m^k \cap E = \varnothing$. Then by the above discussion we see that

$$\left| \sum_{m=1}^{n(k)} \psi(J_m^k) - T^C(f, I_k) \right| < \varepsilon/3N.$$

For $k > N$, clearly $|\sum_{m=1}^{n(k)} \psi(J_m^k)| \leqq O(T^C; f; I_k)$. Thus,

$$\begin{aligned}
&\left| \sum_{k=1}^{p} \psi(J_k) - \sum_{k=1}^{\infty} T^C(f, I_k) \right| \\
&\quad = \left| \sum_{k=1}^{\infty} \sum_{m=1}^{n(k)} \psi(J_m^k) - \sum_{k=1}^{N} T^C(f, I_k) - \sum_{k=N+1}^{\infty} T^C(f, I_k) \right| \\
&\quad \leqq \left| \sum_{k=1}^{N} \left[\sum_{m=1}^{n(k)} \psi(J_m^k) - T^C(f, I_k) \right] \right| + \sum_{k=N+1}^{\infty} \left| \sum_{m=1}^{n(k)} \psi(J_m^k) \right| + \sum_{k=N+1}^{\infty} |T^C(f, I_k)| \\
&\quad < N\varepsilon/3N + 2 \sum_{N+1}^{\infty} O(T^C; f; I_k) < \varepsilon.
\end{aligned}$$

Thus $\int_I \psi$ exists and $\int_I \psi = \sum_1^\infty T^C(f, I_k)$. The continuity of $\int \psi$ is clear.

We now show that (2) implies (1). Let $\delta = \delta(1) > 0$ be such that

$$\left| \sum_{I' \in S_I} \psi(I') - \int_I \psi \right| < 1$$

whenever S_I is a partition of I with $|S_I| < \delta$, and also such that

$$\left| \sum_{I' \in S} \psi(I') - \sum_{I' \in S} \int_{I'} \psi \right| < 1$$

whenever S is a finite family of nonoverlapping subintervals of I with $\sup_{I' \in S} |I'| < \delta$ (see [1], 70).

Choose $N=N(\delta)$ such that $\sum_{N+1}^{\infty}|I_k|<\delta/2$. We shall divide $\{I_k\}_{N+1}^{\infty}$ into two classes, $\mathscr{I}^+$ and $\mathscr{I}^-$, as follows: put I_k in $\mathscr{I}^+$ if there exists $\{I_j^k\}$, with $I_j^k \subseteq I_k$, such that $T^C(f, I_j^k) \to O(T^C; f; I_k)$ and put I_k in $\mathscr{I}^-$ otherwise. Write $\mathscr{I}^+=\{I_{n_i}\}=\{J_i\}$. Choose J_i' with $J_i' \subseteq J_i^0$, such that $O(T^C; f; J_i) - T^C(f, J_i') < 1/2^i$. We shall show that $\sum O(T^C; f; J_i) < \infty$. A similar argument yields the same inequality for the sum of the oscillations over the members of $\mathscr{I}^-$. The choice of J_i' induces a partition of J_i, $\{J_i', J_i'', J_i'''\}$. Append to the family $\{J_i\}$ the family $\{I_1, \ldots, I_n\}$. For each $k=1, \ldots, N$, choose a partition $\{I_1^k, \ldots, I_{n(k)}^k\}$ (enumerated from left to right) of I_k such that $|I_j^k|<\delta$ for all j and such that

$$\left|T^C(f, I_k) - \sum_{2}^{n(k)-1} T^C(f, I_j^k)\right| < 1/2N.$$

This can be done since $T^C(f, I)$ is additive and continuous on I_k for all k. For any positive integer p, consider

$$H = I - \left[\left(\bigcup_1^N I_j\right) \cup \left(\bigcup_1^p J_i\right)\right].$$

If H is nonvoid, it consists of a finite number of intervals. We shall now form a partition S of I. Put $\{J_i', J_i'', J_i'''\}_{i=1}^p \cup \{I_1^k, \ldots, I_{n(k)}^k\}_{k=1}^N$ into S. Suppose $I'=[a', b']$ and $I''=[a'', b'']$ are two of these intervals with $b'<a''$ such that none of the above intervals intersects (b', a''). Then, since the sum of the lengths of the contiguous intervals contained in H is $<\delta/2$, it follows that we can partition (b', a'') by a sequence of intervals all of whose end-points are in E, and all of which have length $<\delta/2$. We put this partition into S. We do this for all pairs I' and I'' satisfying the above conditions. We thereby obtain a partition S of I such that $|S|<\delta$. Then

$$\left|\int_I \psi - \sum_{I' \in S} \psi(I')\right| < 1.$$

But

$$\sum_{I' \in S} \psi(I') = \sum_{j=1}^{N} \sum_{k=2}^{n(j)-1} T(f, I_k^j) + \sum_{i=1}^{p} T(f, J_i').$$

Thus

$$\begin{aligned}\sum_{i=1}^{p} T(f, J_j') &\leqq \left|\sum_{j=1}^{N} \sum_{k=2}^{n(j)-1} T(f, I_k^j)\right| + 1 + \left|\int_I \psi\right| \\ &\leqq 1 + \left|\int_I \psi\right| + N/2N + \sum_1^N |T^C(f, I_k)| < 2 + \left|\int_I \psi\right| + \sum_1^N |T^C(f, I_k)|.\end{aligned}$$

Since $O(T^C; f; J_i) - T(f, J_i') < 1/2^i$ for all i,

$$\begin{aligned}\sum_{i=1}^{p} O(T^C; f; J_i) &\leqq \sum_{i=1}^{p} 1/2^i + 2 + \left|\int_I \psi\right| + \sum_1^N |T^C(f, I_k)| \\ &< 3 + \left|\int_I \psi\right| + \sum_1^N |T^C(f, I_k)|.\end{aligned}$$

Since $3+|\int_I \psi|+\sum_1^N |T^C(f, I_k)|$ is an absolute finite constant and p was arbitrary, $\sum_1^\infty O(T^C; f; J_i)<\infty$. As we noted above, a similar argument can be used to establish the result for $\mathscr{I}^-$. Thus (2) implies (1).

The following example illustrates the fact that

$$T^H \subsetneqq T^{H^*} \text{ and } T^C \subsetneqq T^{H^*}.$$

Let

$$\{I_n\} = \left\{\left[\frac{1}{n+1}, \frac{1}{n}\right]\right\} n = 1, 2, \ldots.$$

Define

$$f(x) = x^2 \sin\frac{1}{x^2} \qquad \text{if } 0 < x \leqq \tfrac{1}{2},$$

$$= (\tfrac{1}{4}\sin 4)(1-x) \quad \text{if } \tfrac{1}{2} \leqq x \leqq 1.$$

Extend f by the formula

$$f(x+n) = f(x)$$

for all integers n. Define

$$f_n(x) = \frac{1}{2^{n+1}} f(n(n+1)x) \quad \text{if } x \in I_n,$$

$$= 0 \qquad \text{if } x \notin I_n,$$

and

$$F(x) = \sum_1^\infty f_n(x)$$

for all $x \in I$. $F(x)$ is clearly continuous on I and differentiable a.e. Let $g(x)=F'(x)$. Let $E=\{0\} \cup \{1/n\}$, $n=2, 3, \ldots$. Clearly g is $\mathscr{L}^C$-integrable on I_n for all n. By Theorem 4, to show g is $\mathscr{L}^{H^*}$-integrable on I we need only show that $\sum O(F; I_k)<\infty$. Since $O(F; I_k)<1/2^k$, this is immediate. Thus g is $\mathscr{L}^{H^*}$-integrable. However, since E is the set of $\mathscr{L}$-singular points of g, and since g is not $\mathscr{L}$-integrable on I_n for any n, it follows that g is not $\mathscr{L}^H$-integrable. It is clear that g is not $\mathscr{L}^C$-integrable.

Finally we shall show that $T^{H^*} \subsetneqq T^{CH}$. Let E be the Cantor set. Let $\{I_n\}=\{[a_n, b_n]\}$ be the sequence of intervals contiguous to E. Let $f(x)$ be as in the previous example and

$$F_n(x) = \frac{1}{2^{n+1}} f\left(\frac{2^{n+1}(x-a_n)}{|I_n|}\right) \quad \text{if } x \in I_n,$$

$$= 0 \qquad \text{otherwise},$$

$n=1, 2, \ldots$. Define

$$F(x) = \sum F_n(x).$$

Clearly $g(x)=F'(x)$ exists a.e. Moreover, each point of E is an $\mathscr{L}^C$-singular point of g, and no other point is singular. In addition, $O(F; I_n)<1/2^n$. Thus $\sum O(F; I_n)<\infty$, implying that g is $\mathscr{L}^{CH}$-integrable. However, I_n contains $2^{n+1}+1$ $\mathscr{L}$-singular points, namely $\{a_n+j(|I_n|/2^{n+1})\}$ $j=0, 1, \ldots, 2^{n+1}$. In addition,

$$O\left(F; \left[a_n+j\frac{|I_n|}{2^{n+1}}, a_n+(j+1)\frac{|I_n|}{2^{n+1}}\right]\right) = \frac{1}{2^{n+1}} O(f; [0, 1])$$

for all j and n. Then, if $H=E \cup [\bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2n+1} \{a_n+j(|I_n|/2^{n+1})\}]$, we note that H is closed and that $\sum O(F; J_k)=\infty$, where $\{J_k\}$ is the sequence of intervals contiguous to H. Moreover, if $P\subseteq I$ is a closed set such that g is $\mathscr{L}$-integrable on P and on all $I'\subseteq I$ with $I' \cap P=\varnothing$, then $\sum O(F; I_j')=\infty$, where $\{I_j'\}$ is the sequence of intervals contiguous to P. For, let

$$I_{nj} = [a_n+j(|I_n|/2^{n+1}), a_n+(j+1)(|I_n|/2^{n+1})],$$

$j=0, 1, \ldots, 2^{n+1}-1$, and write $a_{nj}=a_n+j(|I_n|/2^{n+1})$. Let c_{nj} be the midpoint of I_{nj}. Note that F is monotone increasing on $[c_{nj}, a_{n,j+1}]$ and that g is positive on $[c_{nj}, a_{n,j+1}]$. Let $\{I_{nj}^k\}$ be the sequence of intervals contiguous to $P \cup \{c_{nj}, a_{n,j+1}\}$ in $[c_{nj}, a_{n,j+1}]$. Then either

$$\int_{P\cap[c_{nj},a_{n,j+1}]} g\,dx \geqq \tfrac{1}{8}|\sin 4|$$

or

$$\sum_k O(F; I_{nj}^k) \geqq \tfrac{1}{8}|\sin 4|.$$

If

$$\int_{P\cap[c_{nj},a_{n,j+1}]} g\,dx \geqq \tfrac{1}{8}|\sin 4|$$

for infinitely many pairs n, j, then $\int_P |g|\,dx=\infty$ contradicting the assumption that g is $\mathscr{L}$-integrable on P. Thus, assuming that g is $\mathscr{L}$-integrable on P,

$$\sum_k O(F; I_{nj}^k) \geqq |\sin 4|/8$$

for infinitely many pairs n, j, implying $\sum O(F; I_j')=\infty$. Thus g is not $\mathscr{L}^{H^*}$-integrable although it is $\mathscr{L}^{CH}$-integrable.

References

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